MATHEMATICS

Kernel operators

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Communicated by Prof. A. C. Zaanen at the meeting of April 29, 1978

INTRODUCTION

In this paper we investigate kernel operators (also often called integral operators) on order ideals of measurable functions. In § 2 we prove that the set of all kernel operators with domain an ideal L of measurable functions and with range contained in a similar ideal M is a band in the Riesz space of all order bounded linear operators from L into M. If the order continuous dual of L separates the points of L (which is the case, for example, if L is one of the familiar L_p-spaces; $p \geqslant 1$), then the band of kernel operators is exactly the band generated by the kernel operators of finite rank. This theorem has been proved by G. Ya. Lozanovskiĭ ([8]) for a special case and by A. V. Buhvalov ([2]) in a more general context, but their methods of proof are very different from ours. Their proofs depend on N. Dunford's theorem that any continuous linear operator from an L_1-space into an L_p-space ($p > 1$) is a kernel operator. Lozanovskiĭ used the theorem as originally proved by Dunford ([4]) for Lebesgue measure in the real line; Buhvalov used a more general variant, also valid for non-separable measures, as proved by means of a lifting in the book ([6]) of A. and C. Ionescu-Tulcea. Our method is based on the fundamental result that any positive linear operator majorized by a kernel

* The contents of this paper are derived from the author's doctoral thesis at Leiden University.

operator is itself a kernel operator, the proof of which uses only the Radon-Nikodym theorem. This enables us to derive a simple proof of Dunford's theorem without any separability assumptions on the measures and without using lifting theory. In § 3 we shall use the results in § 2 to derive a simple necessary and sufficient condition for an operator to be a kernel operator. For measure spaces of finite measure the condition states that any order bounded sequence which converges to zero in measure should be transformed into a sequence converging pointwise to zero. This condition has also been derived by A. V. Buhvalov ([2] and [3]) and our proof is similar to his; both proofs are analogous to a proof about bilinear forms by H. Nakano ([13], p. 483, th. 5.2). In § 4 we shall use the necessary and sufficient condition to prove some representation theorems for operators to be kernel operators. As a special case we obtain Dunford's theorem in the general form as mentioned above.

1. PRELIMINARIES

For the definitions and basic properties of Riesz spaces we refer to [10]. For the measure theory we refer to [15]. The measure space (X, Λ, μ) will always be a σ-finite measure space and we shall also assume that the Carathéodory extension procedure has been applied to μ. By $M(X, \mu)$ we shall denote the Dedekind complete (and order separable) Riesz space of all real valued μ-measurable functions on X with identification of functions equal a.e. The set of all positive extended real valued μ-measurable functions on X will be denoted by $P(X, \mu)$. Functions in $P(X, \mu)$ differing only on a μ-null set are also identified. The set $P(X, \mu)$ has a natural lattice ordering.

LEMMA 1.1. *Let $\{f_\tau : \tau \in \{\tau\}\}$ be a set of non-negative functions in $M(X, \mu)$. Then $f_0 = \sup f_\tau$ exists in $P(X, \mu)$ and f_0 is already the supremum of an at most countable subset of the set of all f_τ.*

PROOF. Let $f_{\tau,n} = \inf(f_\tau, n)$ for $n = 1, 2, \ldots$. The supremum $u_n = \sup_\tau f_{\tau,n}$ exists in $M(X, \mu)$ for every n, and u_n is already the supremum of an at most countable subset of $(f_{\tau,n} : \tau \in \{\tau\})$. It follows easily that $f_0 = \sup_n u_n$ is the required supremum.

Let L be an order ideal in $M(X, \mu)$, i.e., if $f \in L$, $g \in M(X, \mu)$ and $|g(x)| \leqslant |f(x)|$ a.e., then $g \in L$. The μ-measurable subset E of X is called an L-zero set if all $f \in L$ vanish a.e. on E. We can remove all L-zero sets simultaneously, since there exists a maximal L-zero set E_∞. (See [10], Exercise 22.11). The set $X - E_\infty$ is called the carrier of L and is denoted by X_L. For any measurable subset E of X_L with $\mu(E) > 0$ there now exists a subset $F \subset E$ such that $\mu(F) > 0$ and $\chi_F \in L$. This implies the existence of a sequence $X_n \uparrow L_L$ such that $\mu(X_n) < \infty$ and $\chi_{X_n} \in L$ for all n (see [15], Ch. 15, § 67, th. 3).

Let (X, Λ, μ) and (Y, Σ, ν) be (σ-finite) measure spaces and let L and M be ideals in $M(Y, \nu)$ and $M(X, \mu)$ respectively. We shall assume throughout this whole paper that the carriers of L and M are equal to Y and X respectively. The zero operator from L into M we shall denote by θ. The linear operator T from L into M is called positive if $Tf \geqslant 0$ for all $f \geqslant 0$ in L and we shall denote this by $T \geqslant \theta$. An operator T is called order bounded if $T = T_1 - T_2$ with $T_1, T_2 \geqslant \theta$. As is well known the set $\mathscr{L}_b(L, M)$ of all order bounded operators from L into M is a Dedekind complete Riesz space.

DEFINITION 1.2. *The linear operator T from L into M is called a kernel operator if there exists a realvalued $\mu \times \nu$-measurable function $T(x, y)$ on $X \times Y$ such that*

(i) $Tf(x) = \int T(x, y) f(y) d\nu(y)$ *a.e. on X for all $f \in L$,*

(ii) $\int |T(x, y) f(y)| d\nu(y) \in M$ *for all $f \in L$.*

Condition (ii) implies that a kernel operator is order bounded as a mapping from L into M. Occasionally we shall also consider operators which only satisfy condition (i), but we shall not call these operators kernel operators.

It is clear that the set of kernel operator from L into M is a linear subspace of $\mathscr{L}_b(L, M)$. It will be the main theorem of the next section that the kernel operators form a band in $\mathscr{L}_b(L, M)$. We conclude the present section with a simple result about kernel operators.

THEOREM 1.3. *Let T be a kernel operator from L into M with kernel $T(x, y)$. Then the following holds.*

(i) *T is a positive operator if and only if $T(x, y) \geqslant 0$ a.e. on $X \times Y$.*

(ii) *$T = \theta$ if and only if $T(x, y) = 0$ a.e. on $X \times Y$.*

PROOF. (i) It is evident that T is positive if $T(x, y) \geqslant 0$ a.e. Conversely, assume that T is positive. As remarked above there exist sequences $Y_n \uparrow Y$ and $X_n \uparrow X$ such that $\nu(Y_n) < \infty$, $\chi_{Y_n} \in L$ and $\mu(X_n) < \infty$ for all n. It is sufficient to prove that $T(x, y) \geqslant 0$ a.e. on $X_n \times Y_n$, so we may just as well assume that $\nu(Y) < \infty$, $\chi_Y \in L$ and $\mu(X) < \infty$. From $\chi_Y \in L$ it follows that

$$\int_Y |T(x, y)| d\nu(y) < \infty$$

a.e. on X, so if we write

$$X_k = (x \in X : \int_Y |T(x, y)| d\nu(y) < k)$$

for $k = 1, 2, \ldots$, then $X_k \uparrow X$. Hence it is sufficient to prove that $T(x, y) \geqslant 0$ a.e. on every $X_k \times Y$. Given the measurable sets $E \subset X_k$ and $F \subset Y$, we have by Tonelli's theorem that

$$\int_{E \times F} T(x, y) d\mu \times \nu = \int_E (T\chi_F)(x) d\mu(x) \geqslant 0.$$

From this it follows easily that the integral of $T(x, y)$ over any $\mu \times \nu$-measurable subset of $X_k \times Y$ is non-negative, and so $T(x, y) \geq 0$ a.e.

(ii) Follows from (i) by applying (i) to T and $-T$.

2. THE BAND OF KERNEL OPERATORS

Let T be a positive linear operator from L into M and let $E \times F$ be a measurable subset of $X \times Y$ such that $\chi_F \in L$ and

$$\int_E (T\chi_F)(x) d\mu(x) < \infty.$$

Furthermore, let Γ be the collection of all sets $A \times B \subset E \times F$ such that A is μ-measurable and B is ν-measurable. Then the following lemma holds.

LEMMA 2.1. *$\lambda(A \times B) = \int_A T(\chi_B) d\mu$ is a finitely additive measure on Γ.*

PROOF. It is evident that $0 \leq \lambda(A \times B) < \infty$ for all $A \times B \in \Gamma$ and also that $A \times B \subset A_1 \times B_1$ implies $\lambda(A \times B) \leq \lambda(A_1 \times B_1)$. The additivity proof seems simple, but there is a small complication, caused by the fact that if $f_0, f_1, \ldots, f_n$ are functions in L such that $f_0(y) = \sum_1^n f_k(y)$ for all $y \in Y$, then $Tf_0(x) = \sum_1^n (Tf_k)(x)$ holds for almost every $x \in X$, and not necessarily for all $x \in X$.

We present the additivity proof. Let $A \times B = \bigcup_1^n A_k \times B_k$ in Γ and such that all $A_k \times B_k$ are mutually disjoint. Then

$$\chi_A(x)\chi_B(y) = \sum_1^n \chi_{A_k}(x)\chi_{B_k}(y)$$

for all $(x, y) \in E \times F$, so if we fix $x_0 \in E$, then

$$\chi_A(x_0)T(\chi_B)(x) = \sum_1^n \chi_{A_k}(x_0)(T\chi_{B_k})(x)$$

holds for almost every $x \in X$. The exceptional null set depends on x_0. More precisely, denoting by $D(x_0)$ the subset of $(1, 2, \ldots, n)$ consisting of those k for which $x_0 \in A_k$, the exceptional null set depends on $D(x_0)$. For x_0 varying in E, there are only finitely many different exceptional null sets, so for almost every $x \in X$ we have

$$\chi_A(x)(T\chi_B)(x) = \sum_1^n \chi_{A_k}(x)(T\chi_{B_k})(x).$$

Hence

$$\int_A (T\chi_B)(x) d\mu(x) = \sum_1^n \int_{A_k} (T\chi_{B_k})(x) d\mu(x).$$

In other words, $\lambda(A \times B) = \sum_1^n \lambda(A_k \times B_k)$.

The next theorem is among the most important results in this paper.

THEOREM 2.2. *If $0 \leq S \leq T$ in $\mathscr{L}_b(L, M)$ and T is a kernel operator with kernel $T(x, y)$, then S is a kernel operator with kernel $S(x, y)$ such that $0 \leq S(x, y) \leq T(x, y)$ a.e. on $X \times Y$.*

PROOF. Step 1. Let E and F be measurable subsets of X and Y respectively such that $\chi_F \in L$ and such that

$$\int_{E\times F} T(x,y)d\mu\times\nu = \int_E T\chi_F d\mu < \infty.$$

Then λ, defined for any measurable subset $P \subset E\times F$ by

$$\lambda(P) = \int_P T(x,y)d\mu\times\nu \tag{1}$$

is a σ-additive measure on the σ-algebra of all $\mu\times\nu$-measurable subsets of $E\times F$ such that λ is absolutely continuous with respect to $\mu\times\nu$.

Let Γ be the semi-ring $\{A\times B: A\subset E$ μ-measurable and $B\subset F$ ν-measurable$\}$. On Γ we define

$$\lambda_1(A\times B) = \int_A (S\chi_B)(x)d\mu(x). \tag{2}$$

It follows from the lemma preceding the present theorem that λ_1 is a finitely additive measure on Γ. Since $0\leqslant\lambda_1\leqslant\lambda$ on Γ and λ is σ-additive, the measure λ_1 is actually σ-additive on Γ. We now apply the Carathéodory extension procedure to the measures λ and λ_1; the exterior measures corresponding to λ and λ_1 will be denoted by λ^* and λ_1^* respectively. Obviously we have $0\leqslant\lambda_1^*(P)\leqslant\lambda^*(P)$ for every subset P of $E\times F$. It is also evident from (1) that every $\mu\times\nu$-measurable subset of $E\times F$ is also λ-measurable. We shall prove now that every $\mu\times\nu$-measurable subset P of $E\times F$ is also λ_1-measurable. This is immediate if P is an at most countable intersection of at most countable unions of sets of Γ. Any $\mu\times\nu$-measurable set differs a null set from an intersection of this kind, so it remains to prove that a $\mu\times\nu$-null set is λ_1-measurable. This is easy; if $\mu\times\nu(N)=0$, then $\lambda(N)=0$, so $\lambda_1^*(N)=0$, which shows that N is λ_1-measurable. Note now that, since $0\leqslant\lambda_1(P)\leqslant\lambda(P)$ holds for every $\mu\times\nu$-measurable $P\subset E\times F$ and λ is absolutely continuous with respect to $\mu\times\nu$, the same holds for λ_1. It follows from the Radon-Nikodym theorem that there exists a $\mu\times\nu$-measurable function $S(x,y)\geqslant 0$ on $E\times F$ such that

$$\lambda_1(P) = \int_P S(x,y)d\mu\times\nu$$

for all $\mu\times\nu$-measurable $P\subset E\times F$. In particular, if $A\times B\in\Gamma$, we have

$$\lambda_1(A\times B) = \int_{A\times B} S(x,y)d\mu\times\nu = \int_A \{\int_B S(x,y)d\nu(y)\}d\mu(x).$$

Comparing this with formula (2), we get

$$(S\chi_B)(x) = \int_Y S(x,y)\chi_B(y)d\nu(y) \text{ for almost every } x. \tag{3}$$

Note that (3) holds for every ν-measurable subset B of F.

Step 2. We assume once more that F is a ν-measurable subset of Y such that $\chi_F\in L$, so the function

$$\int_F T(x,y)d\nu(y)$$

is finite a.e. on X. For $k=1, 2, \ldots$, let

$$E_k = (x: k-1\leqslant \int_F T(x,y)d\nu(y) < k).$$

Then X is the union of the disjoint sets E_k. By dividing X into at most countably many sets of finite measure and by step 1 we get that there exists a $\mu \times \nu$-measurable $S_k(x, y) \geqslant 0$ on $E_k \times F$ such that for every measurable $B \subset F$ we have

$$(S\chi_B)(x) = \int_Y S_k(x, y)\chi_B(y)d\nu(y)$$

a.e. on E_k. Defining now $S(x, y)$ on $X \times F$ by $S(x, y) = S_k(x, y)$ on $E_k \times F$ for $k = 1, 2, \ldots$, we get for every measurable $B \subset F$ that

$$(S\chi_B)(x) = \int_Y S(x, y)\chi_B(y)d\nu(y)$$

holds μ- a.e. on X. It follows immediately that if t is a measurable step function on Y vanishing outside F, then

$$(St)(x) = \int_Y S(x, y)t(y)d\nu(y)$$

μ- a.e. on X.

Step 3. Let $Y_n \uparrow Y$ such that $\chi_{Y_n} \in L$ for all n and let $D_1 = Y_1$ and $D_n = Y_n - Y_{n-1}$ for $n = 2, 3, \ldots$. Then Y is the disjoint union of the sets D_n and each χ_{D_n} is in L. Applying the result in step 2 to each $X \times D_n$ separately, we obtain a $\mu \times \nu$-measurable $S(x, y) \geqslant 0$ on $X \times Y$ such that

$$(St)(x) = \int_Y S(x, y)t(y)d\nu(y)$$

holds for each measurable step function t on Y vanishing outside some Y_n.

Step 4. Let $0 \leqslant f_n \uparrow f$ in L. Then by the monotone convergence theorem $Tf_n \uparrow Tf$ in M. Since $\Theta \leqslant S \leqslant T$, it follows that also $Sf_n \uparrow Sf$ in M.

Step 5. Given $0 \leqslant f \in L$, there exists a sequence of measurable step functions t_n $(n = 1, 2, \ldots)$ satisfying $0 \leqslant t_n(y) \uparrow f(y)$ a.e. on Y. It may be assumed that for each n the step function t_n vanishes outside Y_n, where Y_n is the same set as in step 3. Hence, by step 3,

$$(St_n)(x) = \int_Y S(x, y)t_n(y)d\nu(y)$$

holds a.e. on X for each n. Now, the left hand side converges a.e. on X to $(Sf)(x)$ by step 4, and the right hand side converges a.e. on X to the integral of $S(x, y)f(y)$ over Y. Hence for almost every $x \in X$ we have

$$(Sf)(x) = \int_Y S(x, y)f(y)d\nu(y).$$

This shows that S is a kernel operator and by theorem 1.3 the kernel $S(x, y)$ is uniquely determined modulo $\mu \times \nu$-null functions and satisfies $0 \leqslant S(x, y) \leqslant T(x, y)$ a.e. on $X \times Y$.

As noted earlier, the kernel operators from L into M form a linear subspace of $\mathscr{L}_b(L, M)$. Given T and S in $\mathscr{L}_b(L, M)$, the supremum sup (T, S) exists in $\mathscr{L}_b(L, M)$. It is a reasonable conjecture that if T and S are kernel operators with kernels $T(x, y)$ and $S(x, y)$ respectively, then sup (T, S) is also a kernel operator with kernel equal to sup $(T(x, y), S(x, y))$ a.e. Similarly for inf (T, S). In particular, for $S = \Theta$, the conjecture is that

$T^+ = \sup(T, \Theta)$ is a kernel operator with kernel $T^+(x, y) = \sup(T(x, y), 0)$. The proof that this is actually true was given by W. A. J. Luxemburg-A. C. Zaanen ([11]) in 1971. The details of their proof are by no means simple because, although we have

$$T^+u = \sup(Tv : 0 \leqslant v \leqslant u)$$

for every $u \in L^+$, it is not true that the value $(T^+u)(x)$ is for almost every $x \in X$ the supremum of the values $(Tv)(x)$. By means of our last theorem we can give now a brief and transparant proof of the result about $\sup(T, S)$ which was mentioned above. Note that this will imply that if T is a kernel operator with kernel $T(x, y)$, then $|T| = \sup(T, -T)$ is a kernel operator with kernel $|T(x, y)|$.

THEOREM 2.3. *If T and S are kernel operators from L into M with kernels $T(x, y)$ and $S(x, y)$ respectively, then* $\sup(T, S)$ *is also a kernel operator with kernel equal a.e. to* $\sup(T(x, y), S(x, y))$.

PROOF. It is sufficient to present the proof for $S = \theta$. The positive part $T^+(x, y)$ of $T(x, y)$ majorizes $T(x, y)$ as well as the zero function, so the operator T_0 corresponding to $T^+(x, y)$ satisfies $T_0 \geqslant T$ and $T_0 \geqslant \theta$, and hence $T_0 \geqslant T^+ = \sup(T, \theta) \geqslant \theta$.

Since $T_0 \geqslant T^+ \geqslant \theta$ and T_0 is a kernel operator, it follows from the above theorem that T^+ is also a kernel operator. Let $T_1(x, y)$ be the kernel of T^+. On account of $T_0 \geqslant T^+$ the kernels satisfy $T^+(x, y) \geqslant T_1(x, y)$ a.e. On the other hand it follows from $T^+ \geqslant T$ and $T^+ \geqslant \theta$ that $T_1(x, y) \geqslant T(x, y)$ as well as $T_1(x, y) \geqslant 0$ a.e., so $T_1(x, y) \geqslant T^+(x, y)$ a.e. Hence $T_1(x, y) = T^+(x, y)$ a.e., i.e., $T^+ = T_0$. This shows that T^+ has $T^+(x, y)$ as kernel.

THEOREM 2.4. *The kernel operators from L into M form a band in $\mathscr{L}_b(L, M)$.*

PROOF. From theorems 2.2 and 2.3 it follows that the kernel operators from L into M form an (order) ideal in $\mathscr{L}_b(L, M)$. Let $\Theta \leqslant T_\tau \uparrow T$ in $\mathscr{L}_b(L, M)$, where all T_τ are kernel operators with kernel $T_\tau(x, y)$. For the proof that the kernel operators form a band, we have to show now that T is a kernel operator. The system of functions $(T_\tau(x, y) : \tau \in \{\tau\})$ is directed upwards in $M(X \times Y, \mu \times \nu)$. According to lemma 1.1 the function $T(x, y) = \sup T_\tau(x, y)$ exists in $P(X \times Y, \mu \times \nu)$ and there exists an increasing subsequence $(T_n(x, y) : n = 1, 2, \ldots)$ such that $0 \leqslant T_n(x, y) \uparrow T(x, y)$ holds $\mu \times \nu$-a.e. Let T_n be the operator corresponding to the kernel $T_n(x, y)$. Now choose f such that $0 \leqslant f \in L$. Since $Tf = \sup T_\tau f$ holds in M, we have $(Tf)(x) \geqslant (T_n f)(x)$ a.e. on X. Observing that

$$(T_n f)(x) = \int_Y T_n(x, y) f(y) d\nu(y) \uparrow \int_Y T(x, y) f(y) d\nu(y)$$

for almost every x, we find therefore that

$$(Tf)(x) \geqslant \int_Y T(x,y)f(y)d\nu(y).$$

a.e. on X. Let us denote the function on the right by $g(x)$. Then for almost every $x \in X$ we have

(1) $\quad 0 \leqslant g(x) \leqslant (Tf)(x),$

so it follows already that $g(x)$ is finite for almost every x. Since $T(x,y) \geqslant T_\tau(x,y)$ for all τ, we have

(2) $\quad g \geqslant \sup_\tau T_\tau f = Tf$ in M.

Combining (1) and (2), we get $g = Tf$, i.e., $Tf(x) = \int_Y T(x,y)f(y)d\nu(y)$ for almost every x. To conclude the proof, we show that $T(x,y) \in M(X \times Y, \mu \times \nu)$, i.e., we show that $T(x,y)$ is finite a.e. It is sufficient to show finiteness on $X \times E$, where E is a subset of Y such that $\chi_E \in L$. Note first that in this case

(3) $\quad (T\chi_E)(x) = \int_E T(x,y)d\nu(y) < \infty$

for almost every x. Assume now that $F \subset X \times E$ is a set of positive measure such that $T(x,y) = \infty$ on F. Writing $F_x = \{y: (x,y) \in F\}$ for all $x \in X$, the set F_x is a subset of E and it follows from $\mu \times \nu(F) > 0$ that there exists a μ-measurable subset X_0 of X such that $\mu(X_0) > 0$ and $\nu(F_x) > 0$ for all $x \in X_0$. Then

$$\int_{F_x} T(x,y)d\nu(y) = \infty$$

for all $x \in X_0$, so

$$(T\chi_E)(x) \geqslant \int_{F_x} T(x,y)d\nu(y) = \infty$$

for all $x \in X_0$. This contradicts (3). Hence $T(x,y)$ is finite a.e.

We shall now prove that under one extra condition the band of kernel operators may be described much more precisely. Let $L^\wedge$ be the set of all $g \in M(Y,\nu)$ such that

$$\int_Y |f(y)g(y)|d\nu(y) < \infty$$

for every $f \in L$. It is evident that $L^\wedge$ is an ideal in $M(Y,\nu)$. The ideal $L^\wedge$ is sometimes called the Köthe associate space of L. Note that it may happen that $L^\wedge = \{0\}$, take e.g. $L = L_p[0,1]$ $(0 < p < 1)$.

If $g \in L^\wedge$ and $h \in M$, then the $\mu \times \nu$-measurable function $h(x)g(y)$ is obviously the kernel of a kernel operator from L into M. Any finite real linear combination of kernel operators of this type is called a kernel operator of finite rank. The set of all kernel operators of finite rank will be denoted by $L^\wedge \otimes M$. We recall a definition. If D is a non-empty subset of a Riesz space L, then the smallest band in L containing D is called the band generated by D. If the Riesz space L is Archimedean, then the band generated by D is equal to the second disjoint complement D^{dd}

of D (see [10], th. 22.3 (i) $\Rightarrow$ (ii)). In the present case, therefore, $(L^\wedge \otimes M)^{dd}$ is the band generated by $L^\wedge \otimes M$ in $\mathscr{L}_b(L, M)$, and so $(L^\wedge \otimes M)^{dd}$ is included in the band of all kernel operators from L into M. Under a rather natural additional condition the band of kernel operators is exactly $(L^\wedge \otimes M)^{dd}$.

THEOREM 2.5. *If the carrier of $L^\wedge$ is Y, then the set of kernel operators from L into M is $(L^\wedge \otimes M)^{dd}$. In other words, the set of kernel operators is the band generated by the kernel operators of finite rank.*

PROOF. It is sufficient to prove that every positive kernel operator is a member of $(L^\wedge \otimes M)^{dd}$. Let $Y_n \uparrow Y$ such that $\chi_{Y_n} \in L \cap L^\wedge$ for all n and let $X_n \uparrow X$ such that $\chi_{X_n} \in M$. Finally, let T be a positive kernel operator with kernel $T(x, y)$. For $n = 1, 2, \ldots$, we denote by S_n the kernel operator with kernel

$$S_n(x, y) = n\chi_{Y_n}(y)\chi_{X_n}(x),$$

and we define T_n by $T_n = \inf(T, S_n)$. Then T_n is therefore a kernel operator possessing the kernel

$$T_n(x, y) = \min(T(x, y), S(x, y)).$$

It is evident that $S_n \in L^\wedge \otimes M$ for all n and $\theta \leqslant T_n \leqslant S_n$, so $T_n \in (L^\wedge \otimes M)^{dd}$ for all n. We have $T_n(x, y) \uparrow T(x, y)$ a.e. on $X \times Y$, so $T_n \uparrow T$ holds in the Riesz space $\mathscr{L}_b(L, M)$. Hence, since all $T_n \in (L^\wedge \otimes M)^{dd}$, also $T \in (L^\wedge \otimes M)^{dd}$.

We note that a necessary and sufficient condition for the carrier of $L^\wedge$ to be equal to Y is that $L^\wedge$ separates the points of L. Furthermore, there exists a large class of order ideals L in $M(Y, \nu)$ such that $L^\wedge$ separates points of L. These are the normed function spaces (or normed Köthe spaces). If the normed function space is norm complete, the space is called a Banach function space. Familiar examples are the spaces $L_p(Y, \nu)$ $(1 \leqslant p \leqslant \infty)$ and the Orlicz spaces $L_\Phi(Y, \nu)$. It has been proved that if L is an arbitrary normed function space, then L and $L^\wedge$ have the same carrier ([15], § 71, theorem 4).

We conclude this section with a remark about theorem 2.5, additional to the remarks made in the introduction. R. J. Nagel and U. Schlotterbeck investigated in [12] a certain class of operators from a Banach lattice L into a Banach lattice M, which under certain conditions appears to be equal to $(L^* \otimes M)^{dd}$. They remark that one can use their result to characterize the kernel operators on L_p-spaces. H. Schaefer ([14], CH. IV, prop. 9.8) explicitly states that the set of all kernel operators from $L = L_p(Y, \nu)$ into $M = L_q(X, \mu)$, for $1 \leqslant p, q \leqslant \infty$, is equal to $(L^\wedge \otimes M)^{dd}$.

3. A NECESSARY AND SUFFICIENT FOR A KERNEL OPERATOR

LEMMA 3.1. *Let M be an order ideal in $M(X, \mu)$ and let $0 \leqslant f_{n,k} \leqslant f_0$ in M such that for every n we have $\inf_k f_{n,k} = 0$ a.e. on X. Let $E_n = (f_{n,k} : k = 1, 2, \ldots)$*

for every n. Then there exist finite subsets $E_n^1 \subset E_n$ $(n=1, 2, \ldots)$ such that for every m we have $\inf(\bigcup_{n=m}^{\infty} E_n^1)=0$.

PROOF. Let $g_{n,k}=\inf(f_{n,1}, \ldots, f_{n,k})$ for $n, k=1, 2, \ldots$. Then $0 \leqslant g_{n,k} \leqslant f_0$ in M and for every n we have $g_{n,k}(x) \downarrow 0$ as $k \to \infty$ a.e. on X. By the fact that M has the Egoroff property (see section 71 of [10]), it follows that there exists a sequence $h_n \in M$ such that $h_n(x) \downarrow 0$ a.e. and such that there exists a number $k(n)$ with $0 \leqslant g_{n,k(n)} \leqslant h_n$. Writing now

$$E_n^1=(f_{n,1}, \ldots, f_{n,k(n)}),$$

we have for any m that

$$0 \leqslant \inf(\bigcup_{n=m}^{\infty} E_n^1) \leqslant \inf(h_n: n \geqslant m)=0,$$

so $\inf(\bigcup_{n=m}^{\infty} E_m^1)=0$.

Let L and M be order ideals in $M(Y, \nu)$ and $M(X, \mu)$ respectively, exactly as in section 2. We shall assume that Y is the carrier of both L and $L^{\wedge}$. Given the sequence $(f_n: n=1, 2, \ldots)$ of measurable functions in L it is said that f_n star-converges to zero if every subsequence of $(f_n: n=1, 2, \ldots)$ contains a subsequence converging pointwise to zero a.e. on Y. This will be denoted by $f_n \xrightarrow{*} 0$.

LEMMA 3.2. *Let $0 \leqslant u_n \leqslant u$ in L. Then the following conditions are equivalent.*

(a) *$u_n \xrightarrow{*} 0$ as $n \to \infty$.*

(b) *u_n converges to zero in measure on every subset of Y of finite measure.*

(c) *For every $E \subset Y$ such that $\chi_E \in L^{\wedge}$ we have*

$$\int_E u_n d\nu \to 0 \text{ as } n \to \infty.$$

PROOF. The proof that (a) and (b) are equivalent is well known and the proof of (a) $\Leftrightarrow$ (c) is left to the reader.

THEOREM 3.3. *(A. V. Buhvalov [2], [3]). For a positive linear operator T from L into M the following conditions are equivalent.*

(a) *T is a kernel operator.*

(b) *$0 \leqslant u_n \leqslant u \in L$ and $u_n \xrightarrow{*} 0$ implies that $Tu_n(x) \to 0$ a.e. on X.*

PROOF. (a) $\Rightarrow$ (b) Let T be a kernel operator with kernel $T(x, y) \geqslant 0$ and let $0 \leqslant u_n \leqslant u \in L$ with $u_n \xrightarrow{*} 0$ as $n \to \infty$. Since $0 \leqslant u \in L$, there exists a set $X_0 \subset X$ with $\mu(X-X_0)=0$ such that

$$\int_Y T(x_0, y)u(y)d\nu(y) < \infty$$

holds for all $x_0 \in X_0$. If $x_0 \in X_0$, it follows for every n that

$$0 \leqslant \int T(x_0, y)u_n(y)d\nu(y) \leqslant \int T(x_0, y)u(y)d\nu(y) < \infty.$$

It follows now from the dominated convergence theorem that

$$\int T(x_0, y)u_n(y)d\nu(y) \to 0 \text{ as } n \to \infty$$

for all $x_0 \in X_0$, i.e., $Tu_n(x) \to 0$ as $n \to \infty$ a.e. on X.

(b) $\Rightarrow$ (a) Let the positive operator T satisfy condition (b). We have $T = T_1 + T_2$ with T_1 a positive kernel operator and $\theta \leqslant T_2 \in (L^{\sim} \otimes M)^d$. This decomposition exists, since the kernel operators form a (projection) band equal to $(L^{\sim} \otimes M)^{dd}$. It follows from $\theta \leqslant T_2 \leqslant T$ that T_2 also satisfies condition (b). Hence we assume that $\theta \leqslant T \in (L^{\sim} \otimes M)^d$ and we have to prove that if T satisfies (b), then $T = \theta$.

Let $Y_n \uparrow Y$ such that $\chi_{Y_n} \in L \cap L^{\sim}$ for all n. If $0 \leqslant u \in L$, then $(u - u\chi_{Y_n})(y) \to 0$ a.e. on Y, so certainly $(u - u\chi_{Y_n}) \xrightarrow{*} 0$. Hence $T(u - u\chi_{Y_n})(x) \to 0$ a.e. on x, i.e., $T(u\chi_{Y_n})(x) \to (Tu)(x)$ for almost every $x \in X$. It follows that it suffices to prove that $(Tu)(x) = 0$ a.e. on X for all $0 \leqslant u \in L$ such that $u = 0$ outside some fixed Y_n. We fix therefore a natural number n_0 and we assume that $0 \leqslant u \in L$ with $u(y) = 0$ outside Y_{n_0}. Let

$$F_0 = (x\colon (Tu)(x) > 0).$$

By S we denote the finite rank operator with kernel

$$(Tu)(x) \cdot \chi_{Y_{n_0}}(y).$$

Then $\inf(T, S) = \theta$ in $\mathscr{L}_b(L, M)$, in particular $\inf(T, S)(u) = 0$ in M. This means that

$$\inf(Sv + T(u - v)\colon 0 \leqslant v \leqslant u) = 0 \text{ in } M.$$

It follows from the order separability of M that there exists an at most countable set $(v_k\colon 0 \leqslant v_k \leqslant u)$ such that

$$\inf_k (Sv_k + T(u - v_k)\colon 0 \leqslant v_k \leqslant u) = 0 \text{ in } M.$$

From $(Sv_k)(x) \leqslant (Sv_k(x) + T(u - v_k)(x))$ a.e. on X it follows that $\inf_k Sv_k(\in) = 0$ a.e. on X, i.e.,

$$(Tu)(x) \cdot \inf_k \int v_k d\nu = 0$$

a.e. on X, where $\int$ denotes integration over Y_{n_0}. It follows that for every natural number n we have

$$\inf_k ((Sv_k + T(u - v_k))(x)\colon \int v_k d\nu \leqslant 1/n) = 0$$

a.e. on F_0. By lemma 3.1 on double sequences there exists a subsequence $(v_{k_n}\colon n = 1, 2, \ldots)$ such that

(1) $\quad \int v_{k_n} d\nu \to 0$ as $n \to \infty$,

(2) $\quad \inf_{n \geqslant m} T(u - v_{k_n})(x) = 0$ a.e. on E_0 for all m.

It follows from (1) by lemma 3.2 that $v_{k_n} \xrightarrow{*} 0$, so $(Tv_{k_n})(x) \to 0$ a.e. on X, in particular $(Tv_{k_n})(x) \to 0$ a.e. on F_0. On the other hand it follows from

(2) that $Tu(x) = \limsup (Tv_{k_n})(x)$ a.e. on F_0. Hence $(Tu)(x) = 0$ a.e. on F_0, and so $\mu(F_0) = 0$ by the definition of F_0. The final result is that $(Tu)(x) = 0$ a.e. on X. This concludes the proof.

REMARK 1. We extend theorem 3.3 to order bounded operators as follows. Let $0 \leqslant u_n \leqslant u$ and $u_n \xrightarrow{*} 0$ imply that $Tu_n \to 0$. We show that T^+ has the same property. Let $v_n = u - u_n$. Then $0 \leqslant T^+ v_n \leqslant T^+ u$ for all n, so $\limsup T^+ v_n \leqslant T^+ u$. On the other hand, for $0 \leqslant v \leqslant u$ it follows from $v_n \xrightarrow{*} u$ that $\inf(v, v_n) \xrightarrow{*} v$, so $T(\inf(v, v_n)) \to Tv$ by hypothesis. Hence

$$Tv = \lim T(\inf(v, v_n)) \leqslant \liminf T^+ v_n,$$

so $T^+ u = \sup(Tv: 0 \leqslant v \leqslant u) \leqslant \liminf T^+ v_n$. It follows that $T^+ u = \lim T^+ v_n$, so $T^+ u_n \to 0$.

REMARK 2. As Buhvalov remarked in [2], one can drop the condition that the carrier of $L^\wedge$ is equal to Y. Assume T is linear operator from L into M satisfying condition (b) of theorem 3.3. Let $Y_n \uparrow Y$ such that $\chi_{Y_n} \in L$ for all n. For each n, let $L_n = L^\infty(Y_n, \nu)$. Then L_n can be considered as an order ideal in $M(Y, \nu)$ and we can apply theorem 3.3 to the restriction T_n of T to L_n. It is then routine to show that T is a kernel operator, when all T_n are kernel operators.

REMARK 3. From the above remarks it follows that theorem 3.3 gives a characterization of order bounded kernel operators on arbitrary ideals of measurable functions. We now indicate that it also characterizes non-order bounded kernel operators. Let T be a linear operator from L into M which satisfies (b) of theorem 3.3. Then T is an order continuous linear mapping from L into $M(X, \mu)$, the space of all μ-measurable functions. Hence it is order bounded as a mapping from L into $M(X, \mu)$. In a sequel to this paper we shall give a description of the order bounded sets in $M(X, \mu)$, from which this will follow easily. Then we can apply theorem 3.3 (combined with remark 1) for the order ideals L and $M(X, \mu)$ and we then get T as a (in general non-order bounded) kernel operator from L into M.

4. REPRESENTATION OF OPERATORS BY MEANS OF KERNELS

In this section we shall prove, by means of theorem 3.3, that for some classes of operators on Banach function spaces these operators can be represented by kernels. Concerning the terminology of Banach function spaces we refer to [15], except that we shall call the function norm ϱ order continuous, instead of absolutely continuous, if it follows from $0 \leqslant f_n \leqslant f_0$ in L_ϱ and $f_n(x) \to 0$ a.e., that $\varrho(f_n) \to 0$. Furthermore we remark that for the Banach function space L_ϱ the first associate space $L'_\varrho = L_{\varrho'}$ coincides with the space $L_\varrho^\wedge$ of the former sections. Despite remark 3 above, we shall still mean by a kernel operator an order bounded operator. At most places we shall indicate the extensions to the non-order bounded case.

LEMMA 4.1. *If L_ϱ is a Banach function space with order continuous norm, then it follows from $0 \leqslant u_n \leqslant u$ in L_ϱ and $u_n \xrightarrow{*} 0$ that $\varrho(u_n) \to 0$.*

PROOF. Clear.

THEOREM 4.2. *If $L_\varrho = L_\varrho(Y, \nu)$ is a Banach function space with order continuous norm, then every continuous linear operator from L_ϱ into $L_\infty = L_\infty(X, \mu)$ is a kernel operator. In particular, if $1 \leqslant p < \infty$, then every continuous linear operator from $L_p(Y, \nu)$ into $L_\infty(X, \mu)$ is a kernel operator.*

PROOF. It is well known and easy to prove that every continuous linear operator from L_ϱ into L_∞ is order bounded. Let $0 \leqslant u_n \leqslant u$ in L_ϱ such that $u_n \xrightarrow{*} 0$. Then $\varrho(u_n) \to 0$ by the above lemma, so it follows from $|Tu_n(x)| \leqslant \|Tu_n\|_\infty \leqslant \|T\|\varrho(u_n)$ a.e. on X that $Tu(x) \to 0$ a.e. on X. Hence, by theorem 3.3, T is a kernel operator from L_ϱ into L_∞.

Given the Banach function space L_ϱ, the Banach dual of L_ϱ will be denoted by L_ϱ^*. Given $g \in L_\varrho'$ (the first associate space of L_ϱ), we can define $G \in L_\varrho^*$ by

$$G(f) = \int fg d\mu$$

for all $f \in L_\varrho$. It is well known that $\|G\| = \varrho'(g)$ and that we can regard L_ϱ' as a closed linear subspace of L_ϱ^* ([15], theorem 69.3). Let $L_\lambda = L_\lambda(Y, \nu)$ and $L_\varrho = L_\varrho(X, \mu)$ be Banach function spaces and let T be a positive linear operator from L_λ into L_ϱ. Then T is (norm) continuous, so every kernel operator T from L_λ into L_ϱ is continuous. If $T(x, y)$ is the kernel of T, then it is known that the restriction $T^\sim$ of T^* to L_ϱ' is a kernel operator from L_ϱ' into L_λ' having $T^\sim(y, x) = T(x, y)$ as kernel. If T is an order bounded linear operator from L_λ into L_ϱ and if the restriction $T^\sim$ of T^* to L_ϱ' is a kernel operator from L_ϱ' into L_λ', then $(T^\sim)^\sim$ is a kernel operator from L_λ'' into L_ϱ''. But $T^{\sim\sim}$ is an extension of T, so T is then a kernel operator from L_λ into L_ϱ.

THEOREM 4.3. *If $L_\varrho = L_\varrho(X, \mu)$ is a Banach function space such that the norm ϱ has the weak Fatou property and the norm ϱ' is order continuous, then every continuous linear operator from $L_1(Y, \nu)$ into $L_\varrho(X, \mu)$ is a kernel operator.*

PROOF. Let T be a continuous linear operator from $L_1(Y, \nu)$ into $L_\varrho(X, \mu)$. Then the restriction $T^\sim$ of T^* to L_ϱ' is a continuous linear operator from $L_\varrho'(X, \mu)$ into $L_\infty(Y, \nu)$, so on account of the preceding theorem the operator $T^\sim$ is a kernel operator from L_ϱ' into L_∞. By the above remarks

$T^{\sim\sim}$ is a kernel operator from L_1 into L_ϱ'', in particular $T^{\sim\sim}$ is an order bounded operator from L_1 into L_ϱ''. Since ϱ has the weak Fatou property, the second associate space L_ϱ'' has the same elements as L_ϱ, so $T = T^{\sim\sim}$ is an order bounded kernel operator from L_1 into L_ϱ.

We remark that if one is only interested in representation of operators by means of kernels without the order boundedness condition, one can drop the condition that ϱ has the weak Fatou property. But the resulting operator, represented by a kernel, will in general not be order bounded.

COROLLARY 4.4. *(N. Dunford's theorem). For $1 < p \leqslant \infty$ every continuous linear operator from $L_1(Y, \nu)$ into $L_p(X, \mu)$ is a kernel operator.*

Abstract versions of theorems 4.2 and 4.3 have been proved by R. Nagel and U. Schlotterbeck ([12]). A. V. Buhvalov [1] proved a variant of theorem 4.3 by means of vector valued measurable functions. Dunford's theorem (cor. 4.4) was proved by N. Dunford [4] in 1936 for Lebesgue measure. L. V. Kantorovitch and B. Vulikh [7] gave a different proof in 1938. N. Dunford and B. J. Pettis [5] generalized the theorem to the case of separable measures. A. and C. Ionescu Tulcea proved in [6] a vector valued version without separability assumptions by using a lifting. Only recently, in 1974, G. Ya. Lozanovskiĭ [9] published a proof for arbitrary σ-finite measures without using lifting theory.

In a sequel to this paper we shall apply the results of this paper to prove some further representation theorems and to prove some generalizations of theorems about Carleman operators. Moreover we shall give a simple proof for a certain characterization of Hille-Tamarkin operators on L_p-spaces.

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